GENERAL CONSTRUCTION OF LAMBDA CALCULUS MODELS(Lambda Calculus and Computer Science Theory)

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GENERAL CONSTRUCTION OF LAMBDA CALCULUS MODELS

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Abstract

This paper presents a new method of constructing models of λ-calculus. Our method is based upon the connection between cartesian closed categories and models of λ-calculus. We shall show four kinds of the models that can be constructed using the method:

(1) a λ-algebra but not a λ-model, which satisfies the η-rule;

(2) a λ-algebra but not a λ-model, which does not satisfy the η-rule;

(3) an extensional λ-model;

(4) a λ-model but not extensional.

Introduction

As for the type free λ-calculus, three kinds of the models are known up to the present: (1) Term models, (2) D_∞, (3) P_ω, and their variations: see Scott (1972, 1976), Plotkin (1978) and Barendregt (1981). This paper presents a general method of constructing models of λ-calculus.

Many authors pointed out the connection between cartesian closed categories and models of λ-calculus in Lambek (1974, 1980), D.Scott (1980), Koymans (1982), Lambek and P.J.Scott (1982), Adach (1983), Barendregt (1982) and Yokouchi (1983). Let **C** be a cartesian closed category. Furthermore suppose that **C** has an object u and a pair of arrows $\varphi : u \longrightarrow u^u$ and $\psi : u^u \longrightarrow u$ such that $\varphi \circ \psi = id_{u^u}$. Then we can naturally construct a model (**C**) of λ-calculus, precisely speaking λ-algebra Barendregt (1981), from **C** with (u, φ, ψ). Conversely for every λ-algebra $\mathfrak{M}$ there exists a cartesian closed category **C** with (u, φ, ψ) such that $\mathfrak{M} = \mathfrak{M}(\mathbf{C})$. Our method of constructing models of λ-calculus is based upon these results.

First we examine cartesian closed categories with a partially order relation on the set of arrows as an additional structure. We call this kind of cartesian closed categories an <u>order-enriched cartesian closed category</u>. For example, the category PO whose objects are all partially ordered sets, and whose arrows are all monotone functions among them is an order-enriched cartesian closed category. In Wand (1979) a similar concept appears. But it does not discuss cartesian closedness.

We shall introduce a new notion called a <u>retraction map</u>

category, whose arrows are retracts in Scott (1972) with certain properties in an order-enriched cartesian closed category. Let **C** and R be an order-enriched cartesian closed category and its retraction map category, respectively. Then R defines a preorder $\precsim$ on the set of all objects of **C**. Moreover when a $\precsim$ b, R defines an embedding of a into b using a retract from a to b.

Next we shall show that another cartesian closed category **E** called an ε-category can be naturally constructed from **C** with R. Furthermore if R satisfies certain properties, then **E** has an object U and a pair of arrows $\Phi : U \longrightarrow U^U$ and $\Psi : U^U \longrightarrow U$ such that $\Phi \circ \Psi = id_{U^U}$. From **E** with (U, Φ, Ψ) a λ-algebra can be constructed.

The above are outlines of our method. We shall show four examples of the models that can be constructed using the method.

(1) $\mathfrak{M}_1^*$: a λ-algebra but not a λ-model, which satisfies the η-rule $\lambda x.Mx = M$;

(2) $\mathfrak{M}_2^*$: a λ-algebra but not a λ-model, which does not satisfy the η-rule;

(3) $\mathfrak{M}_1^\circ$: an extensional λ-model;

(4) $\mathfrak{M}_2^\circ$: a λ-model not extensional.

The definitions of λ-algebras, λ-models (weakly extensional λ-algebras) and extensional λ-models appear in Barendregt (1981). Also see Hindley and Longo (1980) and Meyer (1982). It is known that the closed term model is a λ-algebra but not a λ-model, which follows from the fact by Plotkin (1974): the λ-calculus is ω-incomplete. While, the above $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ are constructed independently of the fact.

I. The Framework of the Theory

1. Order-enriched cartesian closed categories

<u>Definition</u>. (Cartesian closed category)

A category C is called a <u>cartesian closed category</u> if C satisfies the following conditions:

(1) C has an object 1 called terminal and an arrow $!_a : a \longrightarrow 1$. Moreover, for each object a, C has only $!_a$ as an arrow from a to 1.

(2) For every pair of objects a and b, C has an object $a \times b$ called product and two arrows $\pi_1^{a,b} : a \times b \longrightarrow a$ and $\pi_2^{a,b} : a \times b \longrightarrow b$. Moreover for every pair of arrows $f : c \longrightarrow a$ and $g : c \longrightarrow b$, C has an arrow $\langle f, g\rangle : c \longrightarrow a \times b$ such that $\pi_1^{a,b} \circ \langle f, g\rangle = f$, $\pi_2^{a,b} \circ \langle f, g\rangle = g$ and $\langle \pi_1^{a,b} \circ h, \pi_2^{a,b} \circ h\rangle = h$ for any $h : c \longrightarrow a \times b$.

(3) For every pair of objects a and b, C has an object b^a called exponential and an arrow $ev^{a,b} : b^a \times a \longrightarrow b$. Moreover for every arrow $f : c \times a \longrightarrow b$, C has an arrow $f^\wedge : c \longrightarrow b^a$ such that $ev^{a,b} \circ \langle f^\wedge \circ \pi_1^{c,a}, \pi_2^{c,a}\rangle = f$ and $(ev^{a,b} \circ \langle h \circ \pi_1^{c,a}, \pi_2^{c,a}\rangle)^\wedge = h$ for any $h : c \longrightarrow b^a$.

In this paper, whenever a cartesian closed category is mentioned, its structure (i.e. 1, $!_{(-)}$, $(-)\times(-)$, $\pi_1^{(-),(-)}$, $\pi_2^{(-),(-)}$, $\langle -,-\rangle$, $(-)^{(-)}$, $ev^{(-),(-)}$, $(-)^\wedge$) is determined. We represent $\langle f \circ \pi_1^{a,b}, g \circ \pi_2^{a,b}\rangle$ as $f \times g$, where $f : a \longrightarrow a'$ and $g : b \longrightarrow b'$.

Definition. (Order enriched cartesian closed category)

An order enriched cartesian closed category is a cartesian closed category **C** with additional structures $(\mathbf{C}(a, b), \leq)$ for each pair of objects a and b in **C**, where $\mathbf{C}(a, b) = \{ f \mid f : a \longrightarrow b \text{ in } \mathbf{C} \}$ and $\leq$ is a partially order relation on $\mathbf{C}(a, b)$. Furthermore $(\mathbf{C}(a, b), \leq)$ must satisfy the following conditions.

(1) If $f \leq f'$ and $g \leq g'$, then $g \circ f \leq g' \circ f'$.

(2) If $f \leq f'$ and $g \leq g'$, then $\langle f, g \rangle \leq \langle f', g' \rangle$.

(3) If $f \leq f'$, then $f^{\wedge} \leq (f')^{\wedge}$.

We shall give an example of order-enriced cartesian closed category in Section 9.

Remark.

(1) $(h \circ (g \times id_a))^{\wedge} = h^{\wedge} \circ g$,

where $f : a \longrightarrow a'$ and $g : b \longrightarrow b'$;

(2) $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$;

(3) $(f \times g) \circ (f' \times g') = (f \circ f') \times (g \circ g')$.

2. Retraction map categories

Definition. (Retraction pair)

Let **C** be an order-enriched cartesian closed category and let $i : a \longrightarrow b$ and $j : b \longrightarrow a$ be two arrows of **C**. When $j \circ i = id_a$ and $i \circ j \leq id_b$, then (i, j) is called a retraction pair from a to b.

Notation.

Let $(i_a, j_a) : a \longrightarrow a'$ and $(i_b, j_b) : b \longrightarrow b'$ be two retraction pair. Then we define a pair of arrows

$i[j_a, i_b] : b^a \longrightarrow b'^{a'}$ and $j[i_a, j_b] : b'^{a'} \longrightarrow b^a$ by

$$i[j_a, i_b] = (i_b \circ ev^{a,b} \circ (id_{b^a} \times j_a))^\wedge,$$

$$j[i_a, j_b] = (j_b \circ ev^{a',b'} \circ (id_{b'^{a'}} \times i_a))^\wedge.$$

The following figures explain the intuitive meaning of $i = i[j_a, i_b]$ and $j = j[i_a, j_b]$.

$$\begin{tikzcd} a' \arrow[r, "i_b \circ f \circ j_a"] \arrow[d, "j_a"'] & b' \\ a \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow i"] & b \arrow[u, "i_b"'] \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[r, "g"] \arrow[dr, phantom, "\Downarrow j"] & b' \arrow[d, "j_b"] \\ a \arrow[u, "i_a"] \arrow[r, "j_b \circ g \circ i_a"'] & b \end{tikzcd}$$

<u>Lemma</u> 1.

Let $(i_a, j_a) : a \longrightarrow a'$, $(i_b, j_b) : b \longrightarrow b'$ and $(i_c, j_c) : c \longrightarrow c'$ be three retraction pairs.

(1) The following are both retraction pairs:

$(i_a \times i_b,\ j_a \times j_b) : a \times b \longrightarrow a' \times b'$,

$(i[j_a, i_b],\ j[i_a, j_b]) : b^a \longrightarrow b'^{a'}$.

(2) $\pi_1^{a,b} = j_a \circ \pi_1^{a',b'} \circ (i_a \times i_b)$,

$\pi_2^{a,b} = j_b \circ \pi_2^{a',b'} \circ (i_a \times i_b)$.

(3) $ev^{a,b} = j_b \circ ev^{a',b'} \circ (i[j_a, i_b] \times i_a)$.

(4) $j[i_a, j_b] \circ (f')^\wedge \circ i_c = (j_b \circ f' \circ (i_c \times i_a))^\wedge$,

where $f' : c' \times a' \longrightarrow b'$.

<u>Proof</u>. We can easily calculate as follows.

(1) $(j_a \times j_b) \circ (i_a \times i_b) = (j_a \circ i_a) \times (j_b \circ i_b) = id_a \times id_b = id_{a \times b}$,

$$(i_a \times i_b) \circ (j_a \times j_b) = (i_a \circ j_a) \times (i_b \circ j_b) \leq id_{a'} \times id_{b'} = id_{a' \times b'},$$

$$\begin{aligned}
& j[i_a, j_b] \circ i[j_a, i_b] \\
&= (j_b \circ ev^{a',b'} \circ (id_{b'a'} \times i_a) \circ (i[j_a, i_b] \times id_a))\hat{} \\
&= (j_b \circ ev^{a',b'} \circ (i[i_a, j_b] \times id_{a'}) \circ (id_{ba} \times i_a))\hat{} \\
&= (j_b \circ i_b \circ ev^{a,b} \circ (id_{ba} \times j_a) \circ (id_{ba} \times i_a))\hat{} \\
&= (ev^{a,b} \circ (id_{ba} \times id_a))\hat{} = id_{ba},
\end{aligned}$$

$$\begin{aligned}
& i[j_a, i_b] \circ j[i_a, j_b] \\
&= (i_b \circ ev^{a,b} \circ (id_{ba} \times j_a) \circ (j[i_a, j_b] \times id_{a'}))\hat{} \\
&= (i_b \circ ev^{a,b} \circ (j[i_a, j_b] \times id_a) \circ (id_{b'a'} \times j_a))\hat{} \\
&= (i_b \circ j_b \circ ev^{a',b'} \circ (id_{b'a'} \times i_a) \circ (id_{b'a'} \times j_a))\hat{} \\
&= (i_b \circ j_b \circ ev^{a',b'} \circ (id_{b'a'} \times (i_a \circ j_a)))\hat{} \\
&\leq (ev^{a',b'} \circ (id_{b'a'} \times id_{a'}))\hat{} = id_{b'a'}.
\end{aligned}$$

(2) Clear.

(3)
$$\begin{aligned}
& j_b \circ ev^{a',b'} \circ (i[j_a, i_b] \times i_a) \\
&= j_b \circ ev^{a',b'} \circ (i[j_a, i_b] \times id_{a'}) \circ (id_{ba} \times i_a) \\
&= j_b \circ i_b \circ ev^{a,b} \circ (id_{ba} \times j_a) \circ (id_{ba} \times i_a) \\
&= ev^{a,b} \circ (id_{ba} \times id_a) = ev^{a,b}.
\end{aligned}$$

(4)
$$\begin{aligned}
& j[i_a, j_b] \circ (f')\hat{} \circ i_c \\
&= (j_b \circ ev^{a',b'} \circ (id_{b'a'} \times i_a) \circ (((f')\hat{} \circ i_c) \times id_a))\hat{} \\
&= (j_b \circ ev^{a',b'} \circ ((f')\hat{} \times id_{a'}) \circ (i_c \times i_a))\hat{} \\
&= (j_b \circ f' \circ (i_c \times i_a))\hat{}.
\end{aligned}$$

□

<u>Definition</u>. (Retraction map category)

Let **C** be an order-enriched cartesian closed category. We call the following category **R** a <u>retraction map category</u> of **C**.

(a) The objects of **R** are just the same as of **C**.

(b) Each arrow of **R** is a retraction pair of **C**.

(c) For each pair of arrows $(i, j) : a \longrightarrow b$ and $(i', j') : b \longrightarrow c$, the composite $(i', j')\circ(i, j)$ is $(i'\circ i, j\circ j') : a \longrightarrow c$.

(d) For each object a, the identity arrow is $(id_a, id_a) : a \longrightarrow a$.

Furthermore R must satisfy the following conditions:

(1) For each pair objects a and b, R has at most one arrow from a to b;

(2) If R has $(i_a, j_a) : a \longrightarrow a'$ and $(i_b, j_b) : b \longrightarrow b'$, then R has $(i_a \times i_b, j_a \times j_b) : a \times b \longrightarrow a' \times b'$ and $(i[j_a, i_b], j[i_a, j_b]) : b^a \longrightarrow b'^{a'}$.

<u>Notation</u>.

Let **C** be an order enriched cartesian closed category and let **R** be a retraction map category of **C**.

(1) We write $a \lesssim b$ in **R** or $a \lesssim b$ (i, j) in **R**, when **R** has an arrow $(i, j) : a \longrightarrow b$.

(2) For each pair of arrows $f : a \longrightarrow b$ and $g : a' \longrightarrow b'$ in **C**, we write $f \precsim g$ in **R**, when $a \lesssim a'$ (i_a, j_a) in **R**, $b \lesssim b'$ (i_b, j_b) in **R** and $f \leq j_b \circ g \circ i_a$.

We sometimes omit 'in **R**', when no confusion arises.

Note that the above relations $\lesssim$ and $\precsim$ are reflexive and transitive (i.e. preorders), and note that

$$f \leq j_b \circ g \circ i_a \quad \text{iff} \quad i_b \circ f \circ j_a \leq g.$$

3. $\mathcal{E}$-Categories

Let **C** be an order enriched cartesian closed category and let

R be a retraction map category of **C**. We will construct a new cartesian closed category **E** called an $\mathcal{E}$-<u>category</u> of **C** with respect to **R**.

(Object) The objects of **R** are nonempty subsets A of the set of all the objects in **C** that satisfy the condition: for every pair of objects a and b in A, there exists some object c in A such that $a \lesssim c$ and $b \lesssim c$ in **R**.

(Pre-arrow) Let A and B be two objects of **E**. We call a nonempty subset F of $\{ f \mid f : a \longrightarrow b,\ a \in A \text{ and } b \in B \}$ a <u>pre-arrow</u> from A to B, if F satisfies the following conditions:

(1) For every pair of f, g F, F contains h such that $f \underset{\sim}{\sqsubset} h$ and $g \underset{\sim}{\sqsubset} h$;

(2) If $f : a \longrightarrow b \in F$, $a' \in A$ and $a \lesssim a'$ in **R**, then there exist $b' \in B$ and $g : a' \longrightarrow b' \in F$ such that $b \lesssim b'$ in **R** and $f \underset{\sim}{\sqsubset} g$;

(3) If $f : a \longrightarrow b \in F$, $b' \in B$ and $b \lesssim b'$ in **R**, then there exist $a' \in A$ and $h : a' \longrightarrow b' \in F$ such that $a \lesssim a'$ in **R** and $f \underset{\sim}{\sqsubset} h$.

<u>Notation</u>.

Let F and G be two pre-arrows from A to B in **E**. We define

(1) $F \lesssim G$ iff $(\forall f \in F)(\exists g \in G)(\ f \underset{\sim}{\sqsubset} g\ )$,

(2) $F \sim G$ iff $F \lesssim G$ and $G \lesssim F$.

(Arrow) The arrows of **E** are equivalence classes of pre-arrows with respect to $\sim$. For each pre-arrow F, we represent the equivalence class containing F as [F].

Notation.

Let $F : A \longrightarrow B$ be a pre-arrow and let $a \in A$ and $b \in B$. Then we define

(1) $F(a, b) = \{ f \mid f : a \longrightarrow b \text{ and } f \in F \}$,

(2) $F^* = \{ \{ g \mid g : a \longrightarrow b \text{ and } (\exists f \in F)(g \precsim f) \}$
$\mid a \in A \text{ and } b \in B \}$.

Remark.

(1) Let F be a pre-arrow. Then F^* is also a pre-arrow and $F^* \sim F$.

(2) Let $F, G : A \longrightarrow B$ be a pair of pre-arrows. Then $F \lesssim G$ iff $F^* \subseteq G^*$.

(Composition) Let $F : A \longrightarrow B$ and $G : B \longrightarrow C$ be two pre-arrows of $\mathbf{E}$. We define

$$G \circ F = \{ g \circ f \mid (\exists a \in A)(\exists b \in B)(\exists c \in C) (f \in F(a, b) \text{ and } g \in G(b, c)) \},$$

which is a pre-arrow from A to C. We difine the composite arrow $[G] \circ [F] : A \longrightarrow C$ of $[F]$ and $[G]$ in $\mathbf{E}$ as $[G] \circ [F] = [G \circ F]$. This definition of composition is guaranteed to be well-defined by the following lemma.

Lemma 2.

Let $F_1, F_2 : A \longrightarrow B$ and $G_1, G_2 : B \longrightarrow C$ be pre-arrows of $\mathbf{E}$. If $F_1 \lesssim F_2$ and $G_1 \lesssim G_2$, then $G_1 F_1 \lesssim G_2 F_2$.

Proof. Suppose that $F_1 \lesssim F_2$ and $G_1 \lesssim G_2$. Let $f_1 : a_1 \longrightarrow b_1 \in F_1$ and $g_1 : b_1 \longrightarrow c_1 \in G_1$. Then there exist $f_2 : a_2 \longrightarrow b_2 \in F_2$ and $g_2 : b_2' \longrightarrow c_2 \in G_2$ such that $f_1 \precsim f_2$ and $g_1 \precsim g_2$. By the definition of objects of $\mathbf{E}$, there exists $b \in B$ such that $b_2 \lesssim b$

and $b_2' \lesssim b$, and by the definition of pre-arrows, there exist $f : a \longrightarrow b \in F_2$ and $g : b \longrightarrow c \in G_2$ such that $f_2 \precsim f$ and $g_2 \precsim g$. Hence $g_1 \circ f_1 \precsim g \circ f$ and $G_1 \circ F_1 \lesssim G_2 \circ F_2$. $\square$

(Idefntity) For each object A of **E**, the identity arrow id_A of **E** is $[\{ id_a \mid a \in A \}]$.

We can easily prove that the above **E** is really a category.

4. Cartesian closedness of $\mathcal{E}$-categories

We shall show that the $\mathcal{E}$-category **E** defined in the previous section is a cartesian closed category.

(Terminal) We define

(1) $1 = \{ 1 \text{ (terminal of } \mathbf{C}) \}$,

(2) $!_A = \{ !_a \mid a \in A \}$

for each object A of **E**. Then 1 is a terminal object of **E** and $!_A = [!_A] : A \longrightarrow 1$ is a unique arrow from A to 1.

(Product) We define

(1) $A \times B = \{ a \times b \mid a \in A \text{ and } b \in B \}$,

(2) $\overline{\Pi}_1^{A,B} = \{ \pi_1^{a,b} \mid a \in A \text{ and } b \in B \}$,

(3) $\overline{\Pi}_2^{A,B} = \{ \pi_2^{a,b} \mid a \in A \text{ and } b \in B \}$,

(4) $\langle F, G\rangle = \{ \langle f, g\rangle \mid (\exists a \in A)(\exists b \in B)(\exists c \in C)$
$(f \in F(c, a) \text{ and } g \in G(c, b)) \}$,

where A and B are objects of **E**, $F : C \longrightarrow A$ and $G : C \longrightarrow B$ are pre-arrows of **E**.

From Lemma 1 (1), A B is an object of **E**. By Lemma 1 (2) it is shown that $\overline{\Pi}_1^{A,B}$ and $\overline{\Pi}_2^{A,B}$ are pre-arrows of **E**. So

$\pi_1^{A,B} = [\,\overline{\pi}_1^{A,B}] : A \quad B \longrightarrow A$ and $\pi_2^{A,B} = [\,\overline{\pi}_2^{A,B}] : A \times B \longrightarrow B$ are arrows of **E**. Clearly $\langle F, G\rangle$ is a pre-arrow. We define $\langle [F], [G]\rangle = [\langle F, G\rangle] : C \longrightarrow A \times B$ for each pair of arrows $[F] : C \longrightarrow A$ and $[G] : C \longrightarrow A$, which is guaranteed to be well-defined by the following lemma.

Lemma 3.

Let $F_1, F_2 : C \longrightarrow A$ and $G_1, G_2 : C \longrightarrow B$ be pre-arrows. If $F_1 \lesssim F_2$ and $G_1 \lesssim G_2$, then $\langle F_1, G_1\rangle \lesssim \langle F_2, G_2\rangle$.

Proof. Suppose that $F_1 \lesssim F_2$ and $G_1 \lesssim G_2$. Let $f_1 : c_1 \longrightarrow a_1 \in F_1$ and $g_1 : c_1 \longrightarrow b_1 \in G_1$. Then, there exist $f_2 : c_2 \longrightarrow a_2 \in F_2$ and $g_2 : c_2' \longrightarrow b_2 \in G_2$ such that $f_1 \sqsubseteq f_2$ and $g_1 \sqsubseteq g_2$. Moreover there exist $c \in C$, $f : c \longrightarrow a \in F_2$ and $g : c \longrightarrow b \in G_2$ such that $c_2 \lesssim c$, $c_2' \lesssim c$, $f_2 \sqsubseteq f$ and $g_2 \sqsubseteq g$. So $\langle f_1, g_1\rangle \sqsubseteq \langle f, g\rangle$, and $\langle F_1, G_1\rangle \lesssim \langle F_2, G_2\rangle$. $\square$

Next we shall show that $A \times B$, $\pi_1^{A,B}$, $\pi_2^{A,B}$ and $\langle [F], [G]\rangle$ satisfy the axioms for products.

(a) For every pair of pre-arrows $F : C \longrightarrow A$ and $G : C \longrightarrow B$,

$$\begin{aligned}&\overline{\pi}_1^{A,B} \circ \langle F, G\rangle \\ &= \{\, \pi_1^{a,b} \circ \langle f, g\rangle \mid a \in A,\ b \in B \\ &\qquad\qquad \text{and } (\exists\, c \in C)(\ f \in F(c, a) \text{ and } g \in G(c, b)\ )\ \} \\ &\sim F.\end{aligned}$$

Hence $\pi_1^{A,B} \circ \langle [F], [G]\rangle = [F]$.

(b) Similarly $\pi_2^{A,B} \circ \langle [F], [G]\rangle = [G]$.

(c) For every pre-arrow $H : C \longrightarrow A \times B$,

$$\langle \overline{\pi}_1^{A,B} \circ H,\ \overline{\pi}_2^{A,B} \circ H\rangle$$

$= \{ \langle \pi_1^{a,b} \circ h,\ \pi_2^{a',b'} \circ h' \rangle \mid a \in A,\ a' \in A,\ b \in B,\ b' \in B,$

$\quad$ and $(\exists c \in C)(\ h \in H(c, a\ b)$ and $h' \in H(c, a'\ b')\ )\ \}$

$\supseteq H$

So $H \lesssim \langle \overline{\pi}_1^{A,B} \circ H,\ \overline{\pi}_2^{A,B} \circ H \rangle$. Conversely if $\langle \pi_1^{a,b} \circ h,\ \pi_2^{a',b'} \circ h' \rangle \in \langle \overline{\pi}_1^{A,B} \circ H,\ \overline{\pi}_2^{A,B} \circ H \rangle$, then there exists $h^* : c^* \longrightarrow a^* \times b^* \in H$ such that $h \lessapprox h^*$ and $h' \lessapprox h^*$, and

$$\langle \pi_1^{a,b} \circ h,\ \pi_2^{a',b'} \circ h' \rangle \lessapprox \langle \pi_1^{a^*,b^*} \circ h^*,\ \pi_2^{a^*,b^*} \circ h^* \rangle = h^*.$$

So $\langle \overline{\pi}_1^{A,B} \circ H,\ \overline{\pi}_2^{A,B} \circ H \rangle \lesssim H$. Hence $\langle \overline{\pi}_1^{A,B} \circ [H],\ \overline{\pi}_2^{A,B} \circ [H] \rangle = [H]$.

(Exponentiation) We define

(1) $B^A = \{\ b^a \mid a \in A$ and $b \in B\ \}$,

(2) $\overline{ev}^{A,B} = \{\ ev^{a,b} \mid a \in A$ and $b \in B\ \}$,

(3) $F^\wedge = \{\ f^\wedge \mid f \in F\ \}$,

where A and B are objects of **E**, and $F : C \times A \longrightarrow B$ is a pre-arrow of **E**.

From Lemma 1 (1), B^A is an object of **E**. From Lemma 1 (3) it is shown that $ev^{A,B}$ is a pre-arrow from $B^A \times A$ to B. So $ev^{A,B} = [\overline{ev}^{A,B}]$ is an arrow from $B^A \times A$ to B. From the definition of order-enriched cartesian closed categories, it is clear that $F^\wedge$ is a pre-arrow from C to B^A. We define $[F]^\wedge = [F^\wedge] : C \longrightarrow B^A$ for each pre-arrow $F : C \times A \longrightarrow B$, which is guaranteed to be well-defined by the following lemma.

<u>Lemma</u> 4.

Let $F_1, F_2 : C \times A \longrightarrow B$ be a pair of pre-arrows of **E**. If $F_1 \lesssim F_2$, then $(F_1)^\wedge \lesssim (F_2)^\wedge$.

<u>Proof</u>. Suppose that $F_1 \lesssim F_2$. If $k : c \times a \longrightarrow b \in F_1$, then there exists $k' : c' \times a' \longrightarrow b' \in F_2$ such that $k \lessapprox k'$. Namely $k \leq j_b \circ k' \circ (i_c \times i_a)$, where $a \lesssim a'$ (i_a, j_a), $b \lesssim b'$ (i_b, j_b) and

$c \precsim c'$ (i_c, j_c). So $k^\wedge \leq (j_b \circ k' \circ (i_c \times i_a))^\wedge$. From Lemma 1 (4) $j[i_a, j_b] \circ (k')^\wedge \circ i_c = (j_b \circ k' \circ (i_c \times i_a))^\wedge$. Hence $k^\wedge \sqsubseteq (k')^\wedge$ and $(F_1)^\wedge \precsim (F_2)^\wedge$. □

Finally we shall show that B^A, $ev^{A,B}$ and $[F]^\wedge$ satisfy the axioms for exponentiation.

(a) For every $F : C \times A \longrightarrow B$,

$$\overline{ev}^{A,B} \circ \langle F^\wedge \circ \overline{\pi}_1^{C,A}, \overline{\pi}_2^{C,A} \rangle$$
$$= \{ ev^{a,b} \circ \langle f^\wedge \circ \pi_1^{c,a}, \pi_2^{c,a} \rangle \mid a \in A,\ b \in B,\ c \in C \text{ and } f \in F(c \times a, b) \}$$
$$= F.$$

(b) For every $G : C \longrightarrow B^A$,

$$(\overline{ev}^{A,B} \circ \langle G \circ \overline{\pi}_1^{C,A}, \overline{\pi}_2^{C,A} \rangle)^\wedge$$
$$= \{ (ev^{a,b} \circ \langle g \circ \pi_1^{c,a}, \pi_2^{c,a} \rangle)^\wedge \mid a \in A,\ b \in B,\ c \in C \text{ and } g \in G(c, b^a) \}$$
$$= G.$$

Moreover we define a partially order relation $\leq$ on the set of all the arrows in **E** by

$$[F] \leq [G] \quad \text{iff} \quad F \precsim G,$$

for each pair of arrows $[F] : A \longrightarrow B$ and $[G] : A \longrightarrow B$. Then from Lemma 2, Lemma 3 and Lemma 4, **E** with this order relation satisfies the conditions for order-enriched cartesian closed categories.

From the above discussions we can conclude the following theorem.

Theorem 1.

The ε-category of an order enriched cartesian closed category **C** with respect to a retraction map category **R** of **C** is an order-enriched cartesian closed category.

5. Phenomena of type freedom in ε-categories

Let **C**, **R** and **E** be an order enriched cartesian closed category, a retraction map category of **C** and the ε-category of **C** with respect to **R**, respectively. We define a relation $\lesssim$ on the set of all the objects of **E** by

$$A \lesssim B \text{ iff } (\forall a \in A)(\exists b \in B)(\ a \lesssim b \text{ in } \mathbf{R}).$$

When $A \lesssim B$, we define a pair of arrows $[I_{A,B}] : A \longrightarrow B$ and $[J_{A,B}] : B \longrightarrow A$ in **E** by

$$I_{A,B} = \{\ i \mid (\exists a \in A)(\exists b \in B)(\ a \lesssim b\ (i,\ j) \text{ in } \mathbf{R}\ )\ \},$$
$$J_{A,B} = \{\ j \mid (\exists a \in A)(\exists b \in B)(\ a \lesssim b\ (i,\ j) \text{ in } \mathbf{R}\ )\ \}.$$

Proposition 1.

Let $A \lesssim B$.

(1) $[J_{A,B}]\circ[I_{A,B}] = id_A$;

(2) $[I_{A,B}]\circ[J_{A,B}] \leq id_B$;

(3) Moreover if $B \lesssim A$, then

$[I_{A,B}] = [J_{B,A}]$ and $[J_{A,B}] = [I_{B,A}]$.

Proof. (1) Because $id_a \in J_{A,B}\circ I_{A,B}$ for every $a \in A$, $id_A \leq [J_{A,B}]\circ[I_{A,B}]$. Conversely, if $i : a \longrightarrow b \in I_{A,B}$ and $j' : b \longrightarrow a' \in J_{A,B}$, then there is a^* A such that $a \lesssim a^*$ and $a' \lesssim a^*$. By the assumption $A \lesssim B$, there exists $b' \in B$ such that $a^* \lesssim b'$, and there exists $b^* \in B$ such that $a^* \lesssim b' \lesssim b^*$ and

$b \lesssim b^*$. Let $a \lesssim a^*$ (i_a, j_a), $a' \lesssim a^*$ (i_a, j_a), $a^* \lesssim b^*$ (i^*, j^*) and $b \lesssim b^*$ (i_b, j_b) in **R**. Then, $j' \circ i \underset{\sim}{\sqsubset} id_{a^*}$, since $j_a \circ id_{a^*} \circ i_a = j_a \circ j^* \circ i^* \circ i_a = j' \circ j_b \circ i_b \circ i = j' \circ i$. Hence $[J_{A,B}] \circ [I_{A,B}] \leq id_A$.

$$\begin{tikzcd} a^* \arrow[r, "i^*"] & b^* \arrow[r, "j^*"] \arrow[d, shift left, "j_b"] & a^* \arrow[d, "j_a"] \\ a \arrow[u, "i_a"] \arrow[r, "i"'] & b \arrow[u, shift left, "i_b"] \arrow[r, "j'"'] & a' \end{tikzcd}$$

(2) Clear.

(3) If $i : a \longrightarrow b \in I_{A,B}$, then $a \lesssim b$. Since $B \lesssim A$, there exists $c \in A$ such that $b \lesssim c$. If $b \lesssim c$ (i', j') in **R**, then $j' : c \longrightarrow b \in J_{B,A}$. Because $j'\,(i'\,i) = i$, $i \underset{\sim}{\sqsubset} j'$. Hence $I_{A,B} \lesssim J_{B,A}$. Conversely we can similarly prove that $J_{B,A} \lesssim I_{A,B}$. So $[I_{A,B}] = [J_{B,A}]$. It is similar that $[J_{A,B}] = [I_{B,A}]$. $\square$

The following are well-known facts.

Fact 1.

Let **C** be a cartesian closed category and let $\varphi : u \longrightarrow u^u$ and $\psi : u^u \longrightarrow u$ be a pair of arrows of **C** such that $\varphi \circ \psi = id_{u^u}$. Let $\mathcal{M}$ be the generated λ-algebra from **C** with (u, φ, ψ). Then the following are equivalent:

(1) $\mathcal{M}$ satisfies the η-rule $\lambda x.Mx = M$, where M is a term and x is a variable that does not freely occur in M;

(2) $\psi \circ \varphi = id_u$.

In general for every λ-model $\mathcal{M}$ the following are equivalent:

(1) $\mathcal{M}$ is extensional;

(2) $\mathfrak{M}$ satisfies the η-rule.

If $U^U \lesssim U$, then $[J_{U^U,U}]\circ[I_{U^U,U}] = id_{U^U}$ by Proposition 1. Thus we can naturally construct a λ-algebra $\mathfrak{M}$ from **E** with $(U, [J_{U^U,U}], [I_{U^U,U}])$. Moreover if $U \lesssim U^U$, then $[I_{U^U,U}]\circ[J_{U^U,U}] = [J_{U,U^U}]\circ[I_{U,U^U}] = id_U$ by Proposition 1. So from Fact 1, $\mathfrak{M}$ satisfies the η-rule.

6. ε^*-categories

Let **E** be an ε-category of an order-enriched cartesian closed category **C**. Let $F : \{a\} \longrightarrow \{b\}$ be a pre-arrow in **E**, where a and b are two objects of **C**. Moreover assume that **C** has the least upper bound $g : a \longrightarrow b$ of F. Let $G = \{g\} : \{a\} \longrightarrow \{b\}$. Then we hope to identify [F] with [G] in **E**. But this is not generally possible. We intend to modify the definition of ε-categories.

<u>Definition</u>. (Directed set)

Let $(X, \leq)$ is a partially ordered set. In general we say that a nonempty subset Y of X is directed, when for every pair of x and y in Y there exists $z \in Y$ such that $x \leq z$ and $y \leq z$.

<u>Definition</u>. (Complete order-enriched cartesian closed category)

Let **C** be an order-enriched cartesian closed category. We say that **C** is complete, when **C** satisfies the following conditions.

(1) Let a and b be two objects of **C**. If F is a directed set of arrows from a to b, then there exists an arrow $f^\sim$ from a to b which is the least upper bound of F. We represent the least upper bound $f^\sim$ as $\bigsqcup F$.

(2) Let a, b and c be three objects of **C**. If F and G are directed sets of arrows from a to b and from b to c respectively, then $\bigsqcup G \circ \bigsqcup F = \bigsqcup \{ g \circ f \mid f \in F \text{ and } g \in G \}$.

Note that $\{ g \circ f \mid f \in F \text{ and } g \in G \}$ is directed.

We shall modify the definition of $\mathcal{E}$-categories. From a complete order-enriched cartesian closed category **C**, we shall construct a cartesian closed category $\mathbf{E}^*$ called an $\mathcal{E}^*$-category.

The definition of objects, pre-arrows, composition and identity arrows in $\mathbf{E}^*$ are just the same as in $\mathcal{E}$-categories. We only modify the definition of arrows.

Notation.

Let $F : A \longrightarrow B$ be a pre-arrow of $\mathbf{E}^*$, let $a \in A$ and $b \in B$, and let $h : a \longrightarrow b$. When $f \leq h$ for every $f \in F(a, b)$, we write $F(a, b) \leq h$.

Definition.

Let $F, G : A \longrightarrow B$ be a pair of pre-arrows of $\mathbf{E}^*$. We write $F \lessapprox G$, when the following condition is satisfied:

$$(\forall a \in A)(\forall b \in B)(\forall h : a \longrightarrow b)$$
$$(\ G^*(a, b) \leq h \text{ implies } F^*(a, b) \leq h\ ).$$

If $F \lessapprox G$ and $G \lessapprox F$, then we write $F \approx G$.

Remark.

Let $F, G : A \longrightarrow B$ be a pair of arrows in $\mathbf{E}^*$.

(1) If $F \lesssim G$, then $F \lessapprox G$.

(2) $F \lessapprox G$ iff $\bigsqcup F^*(a, b) \leq \bigsqcup G^*(a, b)$ for every pair of $a \in A$ and $b \in B$.

Let A and B be two objects of **E**. The arrows from A to B are equivalence classes of pre-arrows from A to B with respect to $\approx$. We use the same notation [F] as in the definition of ε-categories for representing the equivalence class that contains each pre-arrow F of $\mathbf{E}^*$.

Lemma 5.

Let **C** be a complete order-enriched cartesian closed category and let $\mathbf{E}^*$ be an ε^*-categeory of **C**.

(1) If $F : A \longrightarrow B$ and $G : B \longrightarrow C$ are two pre-arrows of $\mathbf{E}^*$, then

$$\bigsqcup(G \circ F)^*(a, c)$$
$$= \bigsqcup\{\, g \circ f \mid (\exists\, b \in B)(\; f \in F^*(a, b) \text{ and } g \in G^*(b, c)\;)\,\}$$

for any $a \in A$ and $c \in C$.

(2) If $F : C \longrightarrow A$ and $G : C \longrightarrow B$ are two pre-arrows of $\mathbf{E}^*$, then

$$\bigsqcup(\langle F, G\rangle)^*(c, a \times b) = \langle \bigsqcup F^*(c, a),\ \bigsqcup G^*(c, b)\rangle$$

for any $a \in A$, $b \in B$ and $c \in C$.

(3) If $F : C \times A \longrightarrow B$ is a pre-arrow of $\mathbf{E}^*$, then

$$\bigsqcup(F\hat{\ })^*(c, b^a) = (\bigsqcup F\hat{\ }(c\ \ a, b))\hat{\ }$$

for any $a \in A$, $b \in B$ and $c \in C$.

Here $G \circ F$, $\langle F, G\rangle$ and $F\hat{\ }$ means the same pre-arrows as defined in Section 3 and Section 4 as to ε-categories.

Proof. (1) Let $L = (G \circ F)^*(a, c)$ and

$$R = \{\, g \circ f \mid (\exists\, b \in B)(\; f \in F^*(a, b) \text{ and } g \in G^*(b, c)\;)\,\}.$$

Then more generally we can show that $L = R$. It is clear that $R \subseteq L$. Conversely if $h \in (G\ F)^*(a, c)$, then there exist $f : a' \longrightarrow b \in F$ and $g : b \longrightarrow c' \in G$ such that $h \precsim g \circ f$. Let $a \precsim a'\ (i_a, j_a)$ and $c \precsim c'\ (i_c, j_c)$. Then $h \leq j_c \circ g \circ f \circ i_a$, $j_c \circ g \in G^*(a, b)$ and $f \circ i_a \in F^*(b, c)$. Thus $L \subseteq R$.

(2) First we shall show that $(\langle F, G\rangle)^* = \langle F^*, G^*\rangle$. If

$h : c \longrightarrow a \times b \in (\langle F, G\rangle)^*$, then there exist $f : c' \longrightarrow a' \in F$ and $g : c' \longrightarrow b' \in G$ such that $h \lessapprox \langle f, g\rangle$. Let $a \lesssim a'$ (i_a, j_a), $b \lesssim b'$ (i_b, j_b) and $c \lesssim c'$ (i_c, j_c). Then $h \leq (j_a \times j_b)\circ\langle f, g\rangle\circ i_c$, $\pi_1^{a,b}\circ h \leq j_a\circ f\circ i_c$ and $\pi_2^{a,b}\circ h \leq j_b\circ g\circ i_c$. Because $\pi_1^{a,b}\circ h \in F^*$ and $\pi_2^{a,b}\circ h \in G^*$, $h = \langle\pi_1^{a,b}\circ h, \pi_2^{a,b}\circ h\rangle \in \langle F^*, G^*\rangle$. Therefore $(\langle F, G\rangle)^* \subseteq \langle F^*, G^*\rangle$.

Conversely if $k_1 : c \longrightarrow a \in F^*$ and $k_2 : c \longrightarrow b \in G^*$, then there exist $f : c' \longrightarrow a' \in F$ and $g : c' \longrightarrow b' \in G$ such that $k_1 \lessapprox f$ and $k_2 \lessapprox g$. Let $a \lesssim a'$ (i_a, j_a), $b \lesssim b'$ (i_b, j_b) and $c \lesssim c'$ (j_c, j_c). Then $k_1 \leq j_a\circ f\circ i_c$, $k_2 \leq j_b\circ g\circ i_c$ and $\langle k_1, k_2\rangle \leq (j_a \times j_b)\circ\langle f, g\rangle\circ i_c$. Because $\langle k_1, k_2\rangle \lessapprox \langle f, g\rangle \in \langle F, G\rangle$, $\langle k_1, k_2\rangle \in (\langle F, G\rangle)^*$. Thus $\langle F^*, G^*\rangle \subseteq (\langle F, G\rangle)^*$.

Using the above,

$$\begin{aligned}
&\pi_1^{a,b}\circ \bigsqcup(\langle F, G\rangle)^*(c, a\ \ b)\\
&= \bigsqcup\{\, \pi_1^{a,b}\circ h \mid h \in (\langle F, G\rangle)^*(c, a\times b)\,\}\\
&= \bigsqcup\{\, \pi_1^{a,b}\circ\langle f, g\rangle \mid f \in F^*(c, a) \text{ and } g \in G^*(c, b)\,\}\\
&= \bigsqcup F^*(c, a).
\end{aligned}$$

Similarly $\pi_2^{a,b}\circ \bigsqcup(\langle F, G\rangle)^*(c, a\times b) = \bigsqcup G^*(c, b)$.

Hence $\bigsqcup(\langle F, G\rangle)^*(c, a\times b) = \langle \bigsqcup F^*(c, a), \bigsqcup G^*(c, b)\rangle$.

(3) First we shall show that $(F^\wedge)^* = (F^*)^\wedge$. If $h : c \longrightarrow b^a \in (F^\wedge)^*$, then there exists $f : c' \times a' \longrightarrow b'$ F such that $h \lessapprox f^\wedge$. Let $a \lesssim a'$ (i_a, j_a), $b \lesssim b'$ (i_b, j_b) and $c \lesssim c'$ (j_c, i_c). Then $h \leq j[i_a, j_b]\circ f^\wedge\circ i_c = (j_b\circ f\circ(i_c \times i_a))^\wedge$ from Lemma 1 (4). Because $j_b\circ f\circ(i_c \times i_a) \in F^*$, $h \in (F^*)^\wedge$. So $(F^\wedge)^* \subseteq (F^*)^\wedge$.

Conversely if $k : c \times a \longrightarrow b \in F^*$, then there exists $f : c' \times a' \longrightarrow b' \in F$ such that $k \lessapprox f$. Let $a \lesssim a'$ (i_a, j_a), $b \lesssim b'$ (i_b, j_b) and $c \lesssim c'$ (i_c, j_c). Because $k \leq j_b\circ f\circ(i_c \times i_a)$,

$k\hat{} \leq (j_b \circ f \circ (i_c \times i_a))\hat{} = j[i_a, j_b] \circ f\hat{} \circ i_c$. So $k\hat{} \lessapprox f\hat{}$ and $k\hat{} \in (F\hat{})^*$. Therefore $(F^*)\hat{} \subseteq (F\hat{})^*$.

Using the above, we can calculate as follows:

$$
\begin{aligned}
& \bigsqcup(F\hat{})^*(c,\ b^a) \\
&= (ev^{a,b} \circ < \bigsqcup(F\hat{})^*(c,\ b^a) \circ \pi_1^{c,a},\ \pi_2^{c,\ a} >)\hat{} \\
&= (\bigsqcup\{<ev^{a,b} \circ <h \circ \pi_1^{c,a},\ \pi_2^{c,a}> \mid h \in (F\hat{})^*(c,\ b^a)\ \})\hat{} \\
&\qquad\qquad \text{using (1) and (2)} \\
&= (\bigsqcup\{<ev^{a,b} \circ <h \circ \pi_1^{c,a},\ \pi_2^{c,a}> \mid h \in (F^*)\hat{}(c,\ b^a)\})\hat{} \\
&= (\bigsqcup\{<ev^{a,b} \circ <f\hat{} \circ \pi_1^{c,a},\ \pi_2^{c,a}> \mid f \in F^*(c\ \ a,\ b)\})\hat{} \\
&= (\bigsqcup F^*(c \times a,\ b))\hat{}.
\end{aligned}
$$

□

<u>Theorem</u> 2.

Let **C** be a complete order-enriched cartesian closed category. Then an $\mathcal{E}^*$-category of **C** is an order-enriched cartesian closed category.

<u>Proof</u>. It is enough to show that Lemma 2, Lemma 3 and Lemma 4 still hold with respect to $\lessapprox$ on the set of pre-arrows.

(Lemma 2) Let $F_1,\ F_2 : A \longrightarrow B$ and $G_1,\ G_2 : B \longrightarrow C$. If $F_1 \lessapprox F_2$ and $G_1 \lessapprox G_2$, then for every pair of $a \in A$ and $c \in C$

$$
\begin{aligned}
& \bigsqcup(G_1 \circ F_1)^*(a,\ c) \\
&= \bigsqcup\{\ g \circ f \mid (\exists b \in B)(\ f \in F_1^*(a,\ b) \text{ and } g \in G_1^*(b,\ c)\ )\ \} \\
&\qquad\qquad\qquad\qquad \text{by Lemma 5 (1)} \\
&= \{\ \bigsqcup\{\ g \circ f \mid f \in F_1^*(a,\ b) \text{ and } g \in G_1^*(b,\ c)\ \} \mid b \in B\ \} \\
&= \{\ \bigsqcup G_1^*(a,\ b) \circ \bigsqcup F_1^*(b,\ c) \mid b \in B\ \} \\
&\leq \{\ \bigsqcup G_2^*(a,\ b) \circ \bigsqcup F_2^*(b,\ c) \mid b \in B\ \} \\
&= (G_2 \circ F_2)^*(a,\ c).
\end{aligned}
$$

Hence $G_1 \circ F_1 \lessapprox G_2 \circ F_2$.

(Lemma 3) Let $F_1,\ F_2 : C \longrightarrow A$ and $G_1,\ G_2 : C \longrightarrow B$ be pre-arrows.

If $F_1 \lessapprox F_2$ and $G_1 \lessapprox G_2$, then for any $a \in A$, $b \in B$ and $c \in C$,

$$\begin{aligned} &\bigsqcup(\langle F_1, G_1\rangle)^*(c, a\times b) \\ &= \langle \bigsqcup F_1^*(c, a), \bigsqcup G_1^*(c, b)\rangle \qquad \text{by Lemma 5 (2)} \\ &\leq \langle \bigsqcup F_2^*(c, a), \bigsqcup G_2^*(c, b)\rangle \\ &= \bigsqcup(\langle F_2, G_2\rangle)^*(c, a\times b). \end{aligned}$$

Hence $\langle F_1, G_1\rangle \lessapprox \langle F_2, G_2\rangle$.

(Lemma 4) Let $F_1, F_2 : C\times A \longrightarrow B$ be pre-arrows. If $F_1 \lessapprox F_2$, then for any $a \in A$, $b \in B$ and $c \in C$,

$$\begin{aligned} &\bigsqcup(\hat{F_1})^*(c, b^a) = (\bigsqcup F_1^*(c\times a, b))^\wedge \qquad \text{by Lemma 5 (3)} \\ &\leq (\bigsqcup F_2^*(c\times a, b))^\wedge = \bigsqcup(\hat{F_2})^*(c, b^a). \end{aligned}$$

Hence $\hat{F_1} \lessapprox \hat{F_2}$. □

7. Some remarks on $\mathcal{E}$-categories and $\mathcal{E}^*$-categories

Let **E** and **E*** be an $\mathcal{E}$-category and an $\mathcal{E}^*$-category, respectiverly. Similarly to $\mathcal{E}$-categories we define a partial order relation $\leq$ on the set of all arrows of **E*** as follows: for each pair of arrows $[F] : A \longrightarrow B$ and $[G] : A \longrightarrow B$, $[F] \leq [G]$ iff $F \lessapprox G$.

Proposition 2.

Both **E** and **E*** are complete order-enriched cartesian closed categories.

Proof. By Theorem 1 and Theorem 2, we have already proved that **E** and **E*** are order-enriched cartesian closed categories. Now we shall show that both **E** and **E*** are complete.

Let P be a directed set of arrows from A to B. If we define $H = \bigcup\{ F \mid [F] \in P \}$, then H is a pre-arrow from A to B and clearly $\bigsqcup P = [H]$. Let Q be a directed set of arrows from B to C. Then

$$\begin{aligned}\bigsqcup Q \circ \bigsqcup P &= [\ \bigcup \{\ G \circ F \mid [F] \in P \text{ and } [G] \in Q\ \}\ ] \\ &= \bigsqcup \{\ [G \circ F] \mid [F] \in P \text{ and } [G] \in Q\ \} \\ &= \bigsqcup \{\ [G] \circ [F] \mid [F] \in P \text{ and } [G] \in Q\ \}.\end{aligned}$$

The above hold in both **E** and **E***. Hence **E** and **E*** are complete. □

Note that $[H] = [\bigcup \{ \alpha([F]) \mid [F] \in P \}]$ in the proof, where $\alpha([F])$ is an arbitrarily chosen pre-arrow contained in [F].

In Section 5 we have defined a preorder relation $\precsim$ on the set of all objects of **E**, and defined two pre-arrows $I_{A,B} : A \longrightarrow B$ and $J_{A,B} : B \longrightarrow A$, when $A \precsim B$. Similarly, in **E***-categories we define $\precsim$, $I_{A,B}$ and $J_{A,B}$, and use the same notations. Then Proposition 1 is also satisfied in the case of $\mathcal{E}^*$-categories, because $F \precsim G$ implies $F \lessapprox G$.

We define a category **R** whose objects are just the same objects of **E** and whose arrows are $([I_{A,B}], [J_{A,B}]) : A \longrightarrow B$ under the condition $A \precsim B$. Similarly we define a category **R*** with respect to **E***.

<u>Proposition</u> 3.

The above **R** and **R*** are retraction map categories of **E** and **E***, respectively. We call **R** and **R*** <u>generated retraction map categories</u> of **E** and **E***.

<u>Proof</u>. We shall prove the proposition as to **R**. If $A \precsim A'$ and $B \precsim B'$, then $A \times B \precsim A' \times B'$ and

$$\begin{aligned}&[I_{A \times B, A' \times B'}] \\ &= [\{\ i_a \times i_b \mid (\exists a \in A)(\exists a' \in A')(\exists b \in B)(\exists b' \in B) \\ &\qquad (\ a \precsim a'\ (i_a,\ j_a) \text{ and } b \precsim b'\ (i_b,\ j_b)\ )\ \}]\end{aligned}$$

$= [\langle I_{A,A'} \circ \overline{\pi}_1^{A,B},\ I_{B,B'} \circ \overline{\pi}_2^{A,B} \rangle]$

$= [I_{A,A'}] \circ [I_{B,B'}]$.

Similarly $[J_{A\times B, A'\times B'}] = [J_{A,A'}] \times [J_{B,B'}]$.

If $A \lesssim A'$ and $B \lesssim B'$, then $B^A \lesssim B'^{A'}$ and

$[I_{B^A, B'^{A'}}]$

$= [\{\ i[j_a, i_b] \mid (\exists a \in A)(\exists a' \in A)(\exists b \in B)(\exists b' \in B)$
$(\ a \lesssim a'\ (i_a, j_a) \text{ and } b \lesssim b'\ (i_b, j_b)\ )\ \}]$

$= [(I_{B,B'} \circ \overline{ev}^{A,B} \circ \langle \overline{\pi}_1^{B^A, A'},\ J_{A,A'} \circ \overline{\pi}_2^{B^A, A'} \rangle)\hat{}\ ]$

$= i[[J_{A,A'}], [I_{B,B'}]]$.

Similarly $[J_{B^A, B'^{A'}}] = j[[I_{A,A'}], [J_{B,B'}]]$. Hence with Proposition 1, R is indeed a retraction map category of **E**.

The proof of the proposition as to $\mathbf{R}^*$ is just the same. □

Proposition 4.

The ε^*-category $\mathbf{E}^\sim$ of $\mathbf{E}^*$ with respect to the generated retraction map category $\mathbf{R}^*$ of $\mathbf{E}^*$ is equivalent to $\mathbf{E}^*$. Namely there is a pair of functors $K : \mathbf{E}^\sim \longrightarrow \mathbf{E}^*$ and $L : \mathbf{E}^* \longrightarrow \mathbf{E}^\sim$ such that $K\,L$ is naturally isomorphic to the identity functor $I_{\mathbf{E}^*}$ and $L\,K$ is also naturally isomorphic to the identity functor $I_{\mathbf{E}^\sim}$.

Proof. For each object X of $\mathbf{E}^\sim$ we define a pair of pre-arrows τ_X and ρ_X by

$\tau_X = \{\ [I_{A, \cup X}] \mid A \in X\ \} : X \longrightarrow \{\cup X\}$,

$\rho_X = \{\ [J_{A, \cup X}] \mid A \in X\ \} : \{\cup X\} \longrightarrow X$.

Note that $A \lesssim \cup X$ in $\mathbf{E}^*$ for every $A \in X$. Then we define a pair of functors K and L as follows:

$K(X) = \cup X$ for each X of $\mathbf{E}^\sim$,

$$K([P]) = \bigsqcup(\tau_Y \circ P \circ \rho_X) : \bigcup X \longrightarrow \bigcup Y$$

for each arrow $[P] : X \longrightarrow Y$ of $\mathbf{E}^{\sim}$,

$L(A) = \{A\}$ for each object A of $\mathbf{E}^*$,

$$L([F]) = [\{[F]\}] : \{A\} \longrightarrow \{B\}$$

for each arrow $[F] : A \longrightarrow B$ of $\mathbf{E}^*$.

Note that $\tau_Y \circ P \circ \rho_X$ is a pre-arrow from $\{\bigcup X\}$ to $\{\bigcup Y\}$. So $\tau_Y \circ P \circ \rho_X$ is a directed set of arrows from $\bigcup X$ to $\bigcup Y$. And note that $K([P])$ is independent of the choice of representatives contained in the equivalence class $[P]$.

For every object A of $\mathbf{E}^*$, $(K \circ L)(A) = A$. For every arrow $[F] : A \longrightarrow B$ of $\mathbf{E}^*$

$$(K \circ L)([F]) = K([\{[F]\}]) = \bigsqcup(\tau_{\{B\}} \circ \{[F]\} \circ \rho_{\{A\}})$$
$$= [I_{B,B}] \circ [F] \circ [J_{A,A}] = [F].$$

So $K \circ L$ is naturally isomorphic to $I_{\mathbf{E}^*}$.

Conversely we shall show that L K is naturally isomrphic to $I_{\mathbf{E}^{\sim}}$. Let $[P] : X \longrightarrow Y$ be an arrow of $\mathbf{E}^{\sim}$. Because $[\rho_Y] \circ [\tau_Y] = id_Y$ and $(L \circ K)([P]) = [\{\bigsqcup(\tau_Y \circ P \circ \rho_X)\}] = [\tau_Y \circ P \circ \rho_X] = [\tau_Y] \circ [P] \circ [\rho_X]$,

$$[\rho_Y] \circ (L \circ K)([P]) = [\rho_Y] \circ [\tau_Y] \circ [P] \circ [\rho_X] = [P] \circ [\rho_X].$$

Thus we can regard $[\rho_{(-)}]$ as a natural transformation from $L \circ K$ to $I_{\mathbf{E}^{\sim}}$.

$$\begin{tikzcd}
(L \circ K)(X) = \{\bigcup X\} \arrow[r, "{[\rho_X]}"] \arrow[d, "(L \circ K)([P])"'] & X \arrow[d, "{[P]}"] \\
(L \circ K)(Y) = \{\bigcup Y\} \arrow[r, "{[\tau_Y]}"'] & Y
\end{tikzcd}$$

Moreover it can be proved that $[\rho_X] \circ [\tau_X] = id_X$ and

$[\tau_X]\circ[\rho_X] = id_{\{\cup X\}}$. The former is clear. The latter is shown by the following calculation:

$$\begin{aligned}
& \bigsqcup([\tau_X]\circ[\rho_X])(\cup X, \cup Y) \\
&= \bigsqcup(\tau_X \cdot \rho_X) \\
&= \bigsqcup\{ [I_{A,\cup X}\, J_{A,\cup X}] \mid A \in X \} \\
&= [\cup\{ I_{A,\cup X}\circ J_{A,\cup X} \mid A \in X \}]
\end{aligned}$$

by the remark below the proof of Proposition 2

$$\begin{aligned}
&= [\{ i\circ j' \mid (\exists A \in X)(\exists a \in A)(\exists b \in X)(\exists b' \in X) \\
&\qquad (a \lesssim b\ (i, j) \text{ and } b \lesssim b'\ (i', j')) \}] \\
&= [\{ i\circ j' \mid (\exists a \in \cup X)(\exists b \in \cup X)(\exists b' \in \cup X) \\
&\qquad (a \lesssim b\ (i, j) \text{ and } b \lesssim b'\ (i', j')) \}] \\
&= [\{ id_a \mid a \in \cup X \}] \\
&= id_{\cup X}.
\end{aligned}$$

Hence $[\rho_{(-)}]$ is a natural isomorphism from L K to $I_{\mathbf{E}^\sim}$. □

Proposition 4 does not generally hold in ε-categories. In the case of ε-categories, it is not generally satisfied that $\tau_X \cdot \rho_X = id_{\{\cup X\}}$ in the proof of Proposition 4. But it can be proved that $[\tau_{(-)}]$ and $[\rho_{(-)}]$ are natural transformations from $I_{\mathbf{E}^\sim}$ to L∘K and from L∘K to $I_{\mathbf{E}^\sim}$, even though $\mathbf{E}^\sim$ is the ε-category of $\mathbf{E}^*$.

8. λ-algebras and λ-models

We can naturally construct a λ-algebra $\mathcal{M}$ from a cartesian closed category C with an object u and a pair of arrows $\varphi : u \longrightarrow u^u$ and $\psi : u^u \longrightarrow u$ such that $\varphi\circ\psi = id_{u^u}$. Then the following is well-known:

Fact 2.

The following two conditions are equivalent fact.

(1) $\mathfrak{M}$ is a λ-model;

(2) For every pair of arrows $f, g : u \longrightarrow u$, $f \neq g$ iff there is an arrow $h : 1 \longrightarrow u$ such that $f \circ h \neq g \circ h$.

We examine the necessary and sufficient conditions that make the generated λ-algebras from suitable $\mathcal{E}^*$-categories -models.

Theorem 3.

Let $\mathbf{C}^*$ be a complete order-enriched cartesian closed category and let $\mathbf{E}^*$ be an $\mathcal{E}^*$-category of $\mathbf{C}^*$. For every pair of objects A and B of $\mathbf{E}^*$, the following conditions are equivalent.

(1) For every pair of objects $a \in A$ and $b \in B$ and every pair of arrows $f, g : a \longrightarrow b$ in $\mathbf{C}^*$, if $f \not\leq g$, then there exists an arrow $h : 1 \longrightarrow a$ in $\mathbf{C}^*$ such that $f\ h \not\leq g\ h$.

(2) For every pair of pre-arrows $F, G : A \longrightarrow B$, if $F \not\lesssim G$, then there exists a pre-arrow $H : 1 \longrightarrow A$ such that $F \circ H \not\lesssim G \circ H$.

Proof. (1) => (2). Assume that the condition (1) is satisfied. If $F \not\lesssim G$, then there exists $a \in A$, $b \in B$ and an arrow $k : a \longrightarrow b$ such that $G^*(a, b) \leq k$ and $F^*(a, b) \not\leq k$. Because $F^*(a, b) \not\leq k$, there exists $f \in F^*(a, b)$ such that $f \not\leq k$. From the conditior (1), there exists an arrow $h : 1 \longrightarrow a$ such that $f \circ h \not\leq k \circ h$. We define a pre-arrow $H = \{\ i \circ h \mid (\exists a' \in A)(\ a \lesssim a'\ (i, j)\ )\ \}$ from 1 to A. We can show that (a) $(G \circ H)^*(1, b) \leq k \circ h$. Indeed if $k' \in (G \circ H)^*(1, b)$, then there are $a' \in A$, $b' \in B$, $g \in G(a', b')$ such that $a \lesssim a'$, $b \lesssim b'$ and $k' \leq j_b \circ g \circ i_a \circ h$, where $a \lesssim a'\ (i_a, j_a)$ and $b \lesssim b'\ (i_b, j_b)$. Because $j_b \circ g \circ i_a \leq k$ from $G^*(a, b) \leq k$, $k' \leq k \circ h$. So $(G \circ H)^*(a, b) \leq k \circ h$.

Next we shall show that (b) $(F \circ H)^*(1, b) \nleq k \circ h$. From $f \in F^*(a, b)$ there exists $a' \in A$, $b' \in B$ and $f' \in F(a', b')$ such that $a \lesssim a'$, $b \lesssim b'$ and $f \leq j_b \circ f' \circ i_a$, where $a \lesssim a'$ (i_a, j_a) and $b \lesssim b'$ (i_b, j_b). So $f\ h \leq j_b \circ f' \circ i_a \circ h$. Because $f' \in F(a', b')$ and $i_a \circ h \in H(1, a')$, $f \circ h \in (F \circ H)^*(1, b)$. From $f \circ h \nleq k \circ h$, $(F \circ H)^*(1, b) \nleq k \circ h$.

From the above (a) and (b), $F \circ H \not\lessapprox G \circ H$. Hence the condition (2) is satisfied.

(2) => (1). Assume that the condition (2) is satisfied. Suppose that $a \in A$, $b \in B$, f, $g : a \longrightarrow b$ and $f \nleq g$. We define two pre-arrows from A to B:

$$F = \{\ i_b \circ f \circ j_a \mid (\exists a' \in A)(\exists b' \in B)$$
$$(\ a \lesssim a'\ (i_a, j_a) \text{ and } b \lesssim b'\ (i_b, j_b)\ )\ \},$$
$$G = \{\ i_b \circ g \circ j_a \mid (\exists a' \in A)(\exists b' \in B)$$
$$(\ a \lesssim a'\ (i_a, j_a) \text{ and } b \lesssim b'\ (i_b, j_b)\ )\ \}.$$

We shall show that $F \not\lessapprox G$. Because $f \in F^*(a, b)$ and $f \nleq g$, $F^*(a, b) \nleq g$. But $G^*(a, b) \leq g$ since $g' \leq g$ for any $g' \in G^*(a, b)$. So $F \not\lessapprox G$.

From the condition (2), there exists a pre-arrow $H : 1 \longrightarrow A$ such that $F\ H \not\lessapprox G\ H$. So there exist $b'' \in B$ and an arrow $k : 1 \longrightarrow b''$ such that $(G \circ H)^*(1, b'') \leq k$ and $(F \circ H)^*(1, b'') \nleq k$. From $(F \circ H)^*(1, b'') \nleq k$, there are $a' \in A$, $b' \in B$, and an arrow $h' : 1 \longrightarrow a'$ such that $a \lesssim a'$, $b \lesssim b'$, $b'' \lesssim b'$ and $j \circ i_b \circ f \circ j_a \circ h' \nleq k$, where $a \lesssim a'$ (i_a, j_a), $b \lesssim b'$ (i_b, j_b) and $b'' \lesssim b'$ (i, j). But $j \circ i_b \circ g \circ j_a \circ h' \leq k$ because $j \circ i_b \circ g \circ j_a \circ h'$ $(G \circ H)^*(1, b'') \leq k$. So $j \circ i_b \circ f \circ j_a \circ h' \nleq j \circ i_b \circ g \circ j_a \circ h'$ and $f \circ j_a \circ h' \nleq g \circ j_a \circ h'$. If we take $j_a \circ h'$ to h, the condition (1) is satisfied. □

Corollary of Theorem 3.

Let $\mathbf{E}^*$ be an $\mathcal{E}^*$-category of a complete order-enriched cartesian closed category $\mathbf{C}^*$. Assume that $\mathbf{E}^*$ has an object U such that $U^U \lesssim U$. Let $\mathfrak{M}$ be the generated λ-algebra from $\mathbf{E}^*$ with $(U, [I_{U,U^U}], [J_{U,U^U}])$. Then the following conditions are equivalent:

(1) $\mathfrak{M}$ is a λ-model;

(2) For every pair of objects a, b $\in$ U and for every pair of arrows f, g : $a \longrightarrow b$ in $\mathbf{C}^*$, if $f \neq g$, then there exists an arrow h : $1 \longrightarrow a$ in $\mathbf{C}^*$ such that $f \circ h \neq g \circ h$.

II. Examples

9. An example P of order-enried cartesian closed categories and its retraction map category $\mathbf{R}_1$

Let **PO** be the category whose objects are all partially ordered sets and whose arrows are all monotone functions among them. We define an order relation on the sets of arrows as follows: for each pair of arrows f, g : $a \longrightarrow b$, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in a$. Then we can prove that **PO** is an order-enriched cartesian closed category. We shall give a sketch of the proof.

For each pair of ordered sets a and b, we define two ordered sets $a \times b$ and b^a as follows:

(1) $a \times b$ is the Cartesian product with the order relation $\leq$ such that $\langle x, y\rangle \leq \langle x', y'\rangle$ iff $x \leq x'$ and $y \leq y'$;

(2) b^a is the set of all the monotone functions from a to b with the order relation $\leq$ such that $f \leq g$ iff $f(x) \leq g(x)$

for all $x \in a$.

For each pair of objects a and b, we define an arrow $ev^{a,b}$ from $b^a \times a$ to b as follows: $ev^{a,b}(\langle f, x\rangle) = f(x)$ for $\langle f, x\rangle \in b^a \times a$. Clearly $ev^{a,b}$ is monotonic. For each arrow $f : c \times a \longrightarrow b$, we define an arrow $f^\wedge : c \longrightarrow b^a$ by $f^\wedge(z)(x) = f(\langle z, x\rangle)$ for $z \in c$, $x \in a$. Clearly $f^\wedge$ is monotonic because f is so. By simple calculation, it can be shown that $ev^{a,b}$ and $f^\wedge$ satisfy the axioms for cartesian closed categories.

The rest structures are clear.

Next we shall define a full subcategory **P** of **PO**. Let a partially ordered set p with the least element $\perp$ be given. We define the set of objects of **P** by indunction.

(1) 1 (a singleton set) is an object of **P**;

(2) p is an object of **P**;

(3) if a and b are objects of **P**, then $a \times b$ and b^a are also objects of **P**.

The arrows of **P** between each pair of objects are the same as of **PO**. Then **P** is an order enriched cartesian closed category.

We shall define a retraction map category $\mathbf{R}_1$ of **P**. Here we assume that, for every object a of **P**, the structure of a is uniquely determined. Namely the set of objects of **P** is the free algebra generated from $\{1, p\}$ by the binary operators $(-)\times(-)$ and $(-)^{(-)}$. For each object a of **P**, we define a nonnegative integer grade(a) as follows:

$$\mathrm{grade}(1) = 0,\ \mathrm{grade}(p) = 0,$$

$$\mathrm{grade}(a \times b) = \mathrm{grade}(a) + \mathrm{grade}(b) + 1,$$

$$\mathrm{grade}(b^a) = \mathrm{grade}(a) + \mathrm{grade}(b) + 1.$$

By the above assumption, each object of **P** has the unique grade.

For each pair of objects a and b such that grade(a) $\leq$ grade(b), we define arrows from a to b in $\mathbf{R}_1$ by double induction on grade(b) and grade(b)-grade(a) (i.e. transfinite induction on $\omega \cdot$grade(b)+(grade(b)-grade(a))).

(1) $(id_c, id_c) : c \longrightarrow c$ is an arrow of $\mathbf{R}_1$;

(2) If $(i_c, j_c) : c \longrightarrow c'$ and $(i_d, j_d) : d \longrightarrow d'$ are arrows of $\mathbf{R}_1$, then $(i_c \times i_d, j_c \times j_d) : c \times d \longrightarrow c' \times d'$ and $(i[j_c, i_d], j[i_c, j_d]) : d^c \longrightarrow d'^{c'}$ are also arrows of $\mathbf{R}_1$;

(3) If $(i, j) : a \longrightarrow c$ and $(i', j') : c \longrightarrow b$ are arrows of $\mathbf{R}_1$ and grade(a) < grade(c) < grade(b), then $(i' \circ i, j \circ j') : a \longrightarrow b$ is an arrow of $\mathbf{R}_1$;

(4) $(\varphi_0, \psi_0) : p \longrightarrow p^p$ is an arrow of $\mathbf{R}_1$, where $\varphi_0 : p \longrightarrow p^p$ and $\psi_0 : p^p \longrightarrow p$ are arrows of $\mathbf{P}$ defined by $\varphi_0(x)(y) = x$ for $x, y \in p$ and $\varphi_0(f) = f(\perp)$ for $f \in p^p$.

We can easily prove that $\mathbf{R}_1$ satisfies the conditions for retraction map categories.

Now let $\mathbf{E}_1$ be the ε-category of $\mathbf{P}$ with respect to $\mathbf{R}_1$. We define an object U of $\mathbf{E}_1$ by induction as follows: $p \in U$; if $a \in U$ and $b \in U$, then $b^a \in U$. Let $\Phi = [J_{U^U,U}] : U \longrightarrow U^U$ and $\Psi = [I_{U^U,U}] : U^U \longrightarrow U$, which are defined in Section 2.5. Because $U \lesssim U^U$ and $U^U \lesssim U$, $\Phi \circ \Psi = id_{U^U}$ and $\Psi \circ \Phi = id_U$ by Proposition 1.

10. Another retraction map category $\mathbf{R}_2$ of $\mathbf{P}$

We shall construct another retraction map category $\mathbf{R}_2$ of $\mathbf{P}$. Note that every object of $\mathbf{P}$ has the least element since p has the least element. We use the same symbol $\perp$ for representing each least element.

For every object a, there exists a retraction pair $(s[a], t[a]) : a \longrightarrow a \times a$. Indeed if we define s[a] and t[a] by

$s[a](x) = \langle x, \perp \rangle$ for $x \in a$,

$t[a](\langle x, y \rangle) = x$ for $\langle x, y \rangle \in a \times a$,

then $(s[a], t[a]) : a \longrightarrow a \times a$ is really a retraction pair. We choose a retraction pair $(s[a], t[a]) : a \longrightarrow a \times a$ for each a.

Next we define arrows from a to b in R_2 by double induction on grade(b) and grade(b)-grade(a).

(1) $(id_c, id_c) : c \longrightarrow c$ is an arrow of R_2;

(2) If $(i_c, j_c) : c \longrightarrow c'$ and $(i_d, j_d) : d \longrightarrow d'$ are arrows of R_2, then $(i_c\, i_d, j_c\, j_d) : c \times d \longrightarrow c' \times d'$ and $(i[j_c, i_d], j[i_c, j_d]) : d^c \longrightarrow d'^{c'}$ are also arrows of R_2;

(3) If $(i, j) : a \longrightarrow c$ and $(i', j') : c \longrightarrow b$ are arrows of R_2 and grade(a) < grade(c) < grade(b), then $(i' \circ i, j \circ j') : a \longrightarrow b$ is an arrow of R_2;

(4) $(\varphi_0, \psi_0) : p \longrightarrow p^p$ is an arrow of R_2, where φ_0 and ψ_0 are the same as defined in Section 9;

(5) $(s[p], t[p]) : p \longrightarrow p \times p$ and $(s[c^{c'}], t[c^{c'}]) : c^{c'} \longrightarrow c^{c'} \times c^{c'}$ are arrows of R_2 for every $c^{c'}$.

<u>Remark</u>.

(1) If grade(b) < grade(a), R_2 contains no arrow from a to b.

(2) For every object c, R_2 contains only $(id_c, id_c) : c \longrightarrow c$ as an arrow from c to c.

(3) For every arrow (i, j) of R_2, i and j preserves the least element $\perp$.

We can prove that R_2 satisfies the conditions for retraction map categories. We only show the following. The rest are clear.

<u>Lemma</u> 6.

For every pair of objects a and b, R_2 contains at most one arrow from a to b.

<u>Proof</u>. We use double induction on grade(b) and grade(b) - grade(a). When grade(a) = grade(b), thr lemma holds by the above remark (2).

(1) When $a = a_1 \times a_2$ and $b = b_1 \times b_2$, some arrows from a to b may be introduced by the rule (2) and (3). By the indunction hypothesis, the number of arrows introduced by the rule (2) is at most one. When the rule (3) is applied, c must be of the form $c_1 \; c_2$ and R_2 must have $(i_1, j_1) : a_1 \longrightarrow c_1$, $(i_2, j_2) : a_2 \longrightarrow c_2$, $(i_1', j_1') : c_1 \longrightarrow b_1$ and $(i_2', j_2') : c_2 \longrightarrow b_2$. So the introduced arrow is $((i_1' \circ i_1) \times (i_2' \circ i_2), (j_1 \circ j_1') \times (j_2 \circ j_2')) : a_1 \times a_2 \longrightarrow b_1 \times b_2$. By the induction hypothesis, the number of arrows introduced by the rule (3) is also at most one. Moreover the arrows introduced by the rule (2) and by the rule (3) are same.

(2) When $b = b_1 \times b_2$ and a is of the form $(a_2)^{a_1}$ or p, some arrows may be introduced by the rule (3) and (5). If the rule (3) is applied, then the rule (5) can not be applied. We consider the case where more than two arrows are introduced by the rule (3). Namely there exists c and c' such that grade(a) < grade(c) < grade(b), grade(a) < grade(c') < grade(b) and R_2 has four arrows from a to c, from a to c', from a to b and from c' to b. Moreover we can assume that R_2 has the arrows illustrated by the following diagrams. Here c is of the form $(c_2)^{c_1} \times (c_2)^{c_1}$ or $(c_2)^{c_1}$ and c' is of the form $(c_2')^{c_1} \times (c_2')^{c_1}$ or $(c_2')^{c_1}$.

$$
\begin{tikzcd}[column sep=small]
 & b_1 & \\
(c_2)^{c_1} \arrow[ur, "{(i_1,\ j_1)}"] & & (c_2')^{c_1} \arrow[ul, "{(i_1',\ j_1')}"'] \\
 & a \arrow[ul, "{(i,\ j)}"] \arrow[ur, "{(i',\ j')}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & b_2 & \\
(c_2)^{c_1} \arrow[ur, "{(i_2,\ j_2)}"] & & (c_2')^{c_1} \arrow[ul, "{(i_1',\ j_1')}"'] \\
 & a \arrow[ul, "{(i,\ j)}"] \arrow[ur, "{(i',\ j')}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & b_1 \times b_2 & \\
(c_2)^{c_1} \times (c_2)^{c_1} \arrow[ur, "{(i_1 \times i_2,\ j_1 \times j_2)}"] & & (c_2')^{c_1} \times (c_2')^{c_1} \arrow[ul, "{(i_1' \times i_2',\ j_1' \times j_2')}"'] \\
(c_2)^{c_1} \arrow[u, "{(s[(c_2)^{c_1}],\ t[(c_2)^{c_1}])}"] & & (c_2')^{c_1} \arrow[u, "{(s[(c_2')^{c_1}],\ t[(c_2')^{c_1}])}"'] \\
 & a \arrow[ul, "{(i,\ j)}"] \arrow[ur, "{(i',\ j')}"'] &
\end{tikzcd}
$$

By the induction hypothesis, $(i_1,j_1)\circ(i,j) = (i_1',j_1')\circ(i',j')$ and $(i_2,j_2)\circ(i,j) = (i_2',j_2')\circ(i',j')$. We define the following arrows:

$$i_1^* = (i_1 \times i_2)\circ s[(c_2)^{c_1}]\circ i \quad : a \longrightarrow b,$$

$$j_1^* = j\circ t[(c_2)^{c_1}]\circ(j_1 \times j_2) \quad : b \longrightarrow a,$$

$$i_2^* = (i_1' \times i_2')\circ s[(c_2')^{c_1}]\circ i' \quad : a \longrightarrow b,$$

and $j_2^* = j'\circ t[(c_2')^{c_1}]\circ(j_1' \times j_2') \quad : b \longrightarrow a.$

By the definition of $(s[-], t[-])$,

$$i_1^*(x) = \langle i_1(i(x)), \perp\rangle = \langle i_1'(i'(x)), \perp\rangle = i_2(x)$$

for all $x \in a$, and

$$j_1^*(\langle y_1, y_2\rangle) = j(j_1(y_1)) = j'(j_1'(y_1)) = j_2^*(\langle y_1, y_2\rangle)$$

for all $\langle y_1, y_2\rangle \in b$. Hence $(i_1^*, j_1^*) = (i_2^*, j_2^*)$.

(3) When $b = (b_2)^{b_1}$, it is clear. □

Let $\mathbf{E}_2$ be the ε-category of $\mathbf{P}$ with respect to $\mathbf{R}_2$. We define an object V of $\mathbf{E}_2$ as the set of all the objects of $\mathbf{P}$ without 1.

<u>Lemma</u> 7.

For every pair of objects a and $b \in V$, there exists $c \in V$ such that $a \precsim c$ in $\mathbf{R}_2$ and $b \precsim c$ in $\mathbf{R}_2$.

<u>Proof</u>. First note that $p \precsim d$ in $\mathbf{R}_2$ for every $d \in V$, which can be proved by induction on grade(d).

We use induction on grade(a) + grade(b).

(1) When $a = p$, $p \precsim b$ by the above remark.

(2) When $b = p$, it is similar to (1).

(3) When $a = a_1 \times a_2$ and $b = b_1 \times b_2$, by the induction hypothesis, there exist $c_1 \in V$ and $c_2 \in V$ such that $a_1 \precsim c_1$, $b_1 \precsim c_1$, $a_2 \precsim c_2$ and $b_2 \precsim c_2$ in $\mathbf{R}_2$. Thus $a_1 \times a_2 \precsim c_1 \times c_2$ and $b_1 \times b_2 \precsim c_1 \times c_2$ in $\mathbf{R}_2$.

(4) When $a = (a_2)^{a_1}$ and $b = (b_2)^{b_1}$, by the induction hypothesis, there exist $c_1 \in V$ and $c_2 \in V$ such that $a_1 \precsim c_1$, $b_1 \precsim c_1$, $a_2 \precsim c_2$ and $b_2 \precsim c_2$ in $\mathbf{R}_2$. So $(a_2)^{a_1} \precsim (c_2)^{c_1}$ and $(b_2)^{b_1} \precsim (c_2)^{c_1}$.

(5) When $a = (a_2)^{a_1}$ and $b = b_1 \times b_2$, there exist $c_1 \in V$ and $c_2 \in V$ such that $a \precsim c_1$, $b_1 \precsim c_1$, $a \precsim c_2$ and $b_2 \precsim c_2$ in $\mathbf{R}_2$. So $a \times a \precsim c_1 \times c_2$ and $b \precsim c_1 \times c_2$ in $\mathbf{R}_2$. From the rule (5) of the

construction of R_2, $a \lesssim a \times a$ in R_2. Hence $a \lesssim c_1 \times c_2$ in R_2.
(6) When $a = a_1 \times a_2$ and $b = (b_2)^{b_1}$, similar to (5). □

Let $\Phi : V \longrightarrow V^V$ and $\Psi : V^V \longrightarrow V$ be two arrows of $\mathbf{E}_2$ defined by $\Phi = [J_{V^V,V}]$ and $\Psi = [I_{V^V,V}]$. Because $V^V \subseteq V$, $\Phi \circ \Psi = id_{V^V}$ by Proposition 1. But it cannot be generally proved that $\Psi \circ \Phi = id_V$.

Lemma 8.

If $\Psi \circ \Phi \neq id_V$, then p is a singleton set.

Proof. If $id_V \leq \Psi \circ \Phi$, then there exist $j : c \longrightarrow b^a \in J_{V_V,V}$ and $i' : b^a \longrightarrow c' \in I_{V^V,V}$ such that $id_{p \times p} \lessapprox i' \circ j$. Because $p \lesssim c$ and $b^a \lesssim c$, from the definition of R_2, c must be of the form $c_1 \times c_2$ and $b^a \lesssim b^a \times b^a \lesssim c_1 \times c_2 = c$. Also $b^a \times b^a \lesssim c'$. Hence $id_{p \times p} \leq (j^* \times j^*) \circ s[b^a] \circ t[b^a] \circ (i^* \times i^*)$, where $p \lesssim b^a$ (i^*, j^*) in R_2. If we apply $\langle x, y\rangle \in p \times p$ to both the hands of this inequation, $\langle x, y\rangle \leq \langle x, j^*(\bot)\rangle$. Because j^* preserves the least element, $y \leq j^*(\bot) = \bot$ for any $y \in p$. Therefore p must be a singleton set. □

$$
\begin{tikzcd}
c \arrow[d, shift left] \arrow[drr, "j"] & & & & c' \arrow[d, shift left] \\
b^a \times b^a \arrow[u, shift left] \arrow[dr, "{t[b^a]}"] & & & & b^a \times b^a \arrow[u, shift left] \arrow[dd, "j^* \times j^*"] \\
& & b^a \arrow[uurr, "i'"] \arrow[urr, "{s[b^a]}"] & & \\
p \times p \arrow[uu, "i^* \times i^*"] \arrow[rrrr, "id_{p \times p}"'] & & & & p \times p
\end{tikzcd}
$$

11. Two examples of λ-models

If p in **P** is finite, then the set of all arrows in **P** is finite. So **P** is complete and the $\mathcal{E}^*$-categories of **P** with respect to R_1 and R_2 are the same as $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively. We shall examine more general cases.

Let **CPO** be the category whose objects are all complete partially ordered sets and whose arrows are all continuous functions. Here a complete partially ordered set means a partially ordered set X such that every directed subset Z of X has the least upper bound $\bigsqcup Z$ in X. A continuous function from X to Y means a function f such that $f(\bigsqcup Z) = \bigsqcup\{ f(x) \mid x \in X \}$ for every directed subset Z of X. It is well-known that **CPO** is a cartesian closed category. Moreover we can easily show that **CPO** is a complete order-enriched cartesian closed category. Here we define an order relation $\leq$ among arrows of **CPO** as follows: for $f, g : a \longrightarrow b$, $f \leq g$ iff $f(x) \leq g(x)$ for any $x \in a$. Then if F is a directed set of arrows from a to b, then $(\bigsqcup F)(x) = \bigsqcup\{ f(x) \mid f \in F \}$ for every $x \in a$.

We shall define a full subcategory $\mathbf{P}^*$ of **CPO** similar to **P**. Let p be a complete partially ordered set with the least element $\perp$. We define $\mathbf{P}^*$ as the category obtained from **CPO** by restricting the set of objects as follows:

(1) 1 (a singleton set) is an object of $\mathbf{P}^*$;

(2) p is an object of $\mathbf{P}^*$;

(3) if a and b are objects of $\mathbf{P}^*$, then $a \times b$ and b^a are also objects of $\mathbf{P}^*$.

Then $\mathbf{P}^*$ is a complete order-enriched cartesian closed category.

We define two retraction map categories R_1^* and R_2^* of $\mathbf{P}^*$ in the same way as R_1 and R_2 of **P**. The proof that R_1^* and R_2^* are really retraction map categories of P^* is just the same as the proof for R_1 and R_2 of **P**. Let E_1^* and E_2^* be the ε^*-categories of P^* with respect to R_1^* and R_2^*, respectively.

We define an object U^* of E_1^* and V^* of E_2^* as the same as U of E_1 and V of E_2, respectively. It is clear that U^* and V^* are objects of E_1^* and E_2^*. Especially V^* in E_2^* satisfies the same as Lemma 6 as to V in E_2. Because $U^{*U^*} \subseteq U^*$ and $V^{*V^*} \subseteq V^*$, $U^{*U^*} \lesssim U^*$ and $V^{*V^*} \lesssim V^*$. Let $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ be λ-algebras generated from E_1^* with $(U^*, [J_{U^*U^*,U^*}], [I_{U^*U^*,U^*}])$ and from E_2^* with $(V^*, [J_{V^*V^*,V^*}], [J_{V^*V^*,V^*}])$, respectively. Then by Corollary of Theorem 3 $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ are both λ-models.

Moreover E_2^* satisfies the following similar to Lemma 8.

<u>Lemma</u> 9.

If $id_{V^*} \leq [I_{V^*V^*,V^*}]\circ[J_{V^*V^*,V^*}]$, then p is a singleton set.

<u>Proof</u>. Let $I = I_{V^*V^*,V^*}$ and $J = J_{V^*V^*,V^*}$. If $id_{V^*} \leq [I]\circ[J]$, then $id_{p\times p} \leq \bigsqcup(I\circ J)^*(p\times p,\ p\times p)$. By Lemma 5 (1)

$$\begin{aligned}&\bigsqcup(I\circ J)^*(p\times p,\ p\times p)\\ &= \bigsqcup\{\ g\circ f \mid (\exists c \in V^{*V^*})(f \in J^*(p\times p,\ c) \text{ and } g \in I^*(c,\ p\times p)\ \}.\end{aligned}$$

If $g \in I^*(c,\ p\times p)$, then there exist $b^a \in V^{*V^*}$ and $d \in V^*$ such that $b^a \lesssim d$ and $g \sqsubseteq i$, where $b^a \lesssim d\ (i,\ j)$. From the definition of R_2^*, $b^a \lesssim b^a \times b^a \lesssim d$ and $i = i'\circ s[b^a]$, where $b^a \times b^a \lesssim d$ $(i',\ j')$. From the definition of $s[b^a]$ and the remark above Lemma 6, $\pi_2^{p,p}(g(z)) = \perp$ for any $z \in c$. So for any $x \in p$ and $y \in p$,

$$\begin{aligned}y &= (\pi_2^{p,p}\circ id_{p\times p})(\langle x,\ y\rangle)\\ &\leq \bigsqcup\{\ (\pi_2^{p,p}\circ g\circ f)(\langle x,\ y\rangle) \mid\\ &\qquad (\exists c \in V^{*V^*})(\ f \in J^*(p\times p,\ c) \text{ and } g \in I^*(c,\ p\times p)\ )\ \}\end{aligned}$$

$= \perp$.

Therefore p must be a singleton set. $\square$

We assume that p is not a singleton set. Then by Lemma 9 and Fact 1, $\mathfrak{M}_2^*$ is not extensional. While $\mathfrak{M}_1^*$ is extentional because $U^* \lesssim U^{*U^*}$.

12. Two examples of λ-algebras but not λ-models

We will give an example of ε^*-category whose generated λ-algebras are not λ-models using Theorem 3.

First we define another complete order-enriched cartesian closed category **CPO°** similar to **CPO**. The objects of **CPO°** are pairs of complete partially ordered sets a and their subsets a' such that for every directed subset X of a' the least upper bound $\bigsqcup X$ is contained in a'. The arrows from (a, a') to (b, b') in **CPO°** are continuous functions from a to b such that $f(x) \in b'$ for every $x \in a'$. The order relation $\leq$ among the arrows is the same as in **CPO**.

The rest components of **CPO°** are defined as follows:

(1) 1 (terminal of **CPO°**) = (1, 1);

(2) $(a, a') \times (b, b') = (a \times b, a' \times b')$;

(3) $(b, b')^{(a, a')} = (b^a, \alpha)$, where

$$\alpha = \{ f \mid f : a \longrightarrow b \text{ is a continuous function and } (\forall x \in a')(\, f(x) \in b'\,) \};$$

(4) $g \circ f = g \circ f$ (in the usual sense)

for $f : (a, a') \longrightarrow (b, b')$ and $g : (b, b') \longrightarrow (c, c')$;

(5) $id_{(a,a')} = id_a$;

(6) $\pi_1^{(a,a'),(b,b')} = \pi_1^{a,b}$;

(7) $\pi_2^{(a,a'),(b,b')} = \pi_2^{a,b}$;

(8) $\langle f, g\rangle = \langle f, g\rangle$ (in the usual sense);

(9) $ev^{(a,a'),(b,b')} = ev^{a,b}$;

(10) $f^\wedge = f^\wedge$ (in the usual sense).

We can easily show that the above $\mathbf{CPO}^\circ$ is a complete cartesian closed category.

Similarly to $\mathbf{P}^*$ we shall define a full subcategory $\mathbf{P}^\circ$ of $\mathbf{CPO}^\circ$ by restricting objects. Let p be a complete partially ordered set with the least element $\perp$. Let p' be a proper subset of p such that $\perp \in p'$ and for every directed subset X p' the least element $\sqcup X$ is contained in p'. Furthermore we assume that (p, p') satisfies the following ciondition: there exists two distinct continuous functions f and g from p to p such that $f(x) = g(x) \in p'$ for every $x \in p'$. For example, if $p = \{\perp, \top\}$ and $p' = \{\perp\}$, then (p, p') satisfies the above conditions. Here we define $\perp \lneq \top$ in p.

We define the objects of $\mathbf{P}^\circ$ as follows:

(1) (1, 1) is an object of $\mathbf{P}^\circ$;

(2) (p, p') is an object of $\mathbf{P}^\circ$;

(3) if (a, a') and (b, b') are objects of $\mathbf{P}^\circ$, then $(a, a') \times (b, b')$ and $(b, b')^{(a, a')}$ are also objects of $\mathbf{P}^\circ$.

Then clearly P is a complete order-enriched cartesian closed category.

We shall define two retraction map category of $\mathbf{R}_1^\circ$ and $\mathbf{R}_2^\circ$ of $\mathbf{P}^\circ$ similar to $\mathbf{R}_1^*$ and $\mathbf{R}_2^*$ of $\mathbf{P}^*$. Let φ_0, ψ_0, s[a] and t[a] be the same as defined in Section 9 and Section 10. Then they are all arrows in $\mathbf{P}^\circ$. Namely in $\mathbf{P}^\circ$,

$$\varphi_0 : (p,\ p') \longrightarrow (p,\ p')^{(p,\ p')},$$

$$\psi_0 : (p,\ p')^{(p,\ p')} \longrightarrow (p,\ p'),$$

$$s[(a,a')] = s[a] : (a,\ a') \longrightarrow (a,\ a') \times (a,\ a'),$$

$$t[(a,a')] = t[a] : (a,\ a') \times (a,\ a') \longrightarrow (a,\ a').$$

Here note that for every object $(a,\ a')$ of $\mathbf{P}^\circ$, a contains the least element and $\perp \in a'$. Using these four kinds of arrows we define R_1° and R_2° in the same way as in R_1^* and R_2^*.

Let $\mathbf{E}_1^\circ$ and $\mathbf{E}_2^\circ$ be the $\mathcal{E}^*$-categories of $\mathbf{P}^\circ$ with respect to R_1° and R_2°, respectively. We define an object U° in $\mathbf{E}_1$ and V° in $\mathbf{E}_2$ in the same way as U in $\mathbf{E}_1$ and V in $\mathbf{E}_2$. Then V° and U° are indeed objects. Especially R_2° satisfies similar properies to Lemma 8 as to R_2. And clearly $U^{\circ U^\circ} \lesssim U^\circ$ and $V^{\circ V^\circ} \lesssim V^\circ$. Let $\mathcal{M}_1^\circ$ and $\mathcal{M}_2^\circ$ be λ-algebras generated from $\mathbf{E}_1^\circ$ with U° and from $\mathbf{E}_2^\circ$ with V°, respectively.

Here $\mathbf{E}_1^\circ$ and $\mathbf{E}_2^\circ$ have comfortable properties. There exists a pair of arrows $f,\ g : (p,\ p') \longrightarrow (p,\ p')$ such that $f \neq g$ and $f \circ h = g \circ h$ for every $h : (1,\ 1) \longrightarrow (p,\ p')$. Note that h must be a function from 1 to p' and that $p' \subsetneq p$. Because $(p,\ p') \in U$ and $(p,\ p') \in V$, from Corollary of Theorem 3 both $\mathcal{M}_1^\circ$ and $\mathcal{M}_2^\circ$ are not λ-models.

We can prove the following similar to Lemma 6 as to $\mathbf{E}_2^*$: if $id_V \leq [I_{V^\circ V^\circ, V^\circ}]\,[J_{V^\circ V^\circ, V^\circ}]$, then p' is a singleton set. So if we assume that p' is not a singleton set, then from Fact 1 $\mathcal{M}_2^\circ$ does not satisfy the η-rule. While $\mathcal{M}_1^\circ$ satisfies the η-rule, since $U^{\circ} \lesssim U^{\circ U^{\circ}}$.

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